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On some dimensional regularisation for Feynman massless integrals

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Abstract. The question of to what extent the conventional interpolatory function for the Gaussian integral $G(w)$ pre-determines the standard meromorphic structure within the dimensional regularisation, is examined for simplest integrals of perturbation theory. It is found that, although it is possible with some generalisation of $G(w)$, to obtain a meromorphic structure for a simplest one-loop integral, it is not sufficient to ensure that higher-order diagrams also have meromorphic structure. An explicit example of such a case is found. All generalisations of $G(w)$ considered lead to a violation of the gauge invariance of the theory.

1. Introduction

It has been suggested by several authors that a continuation in the number of dimensions may be a convenient regularising technique, especially in the case of gauge theories (Speer and Westerwater 1971, 'tHooft and Veltman 1972, Bollini and Giambiaga 1972, Cicuta and Montaldi 1972, Ashmore 1972, 1973, Butera *et al* 1974, Speer 1974a, b, Vega and Schapošnik 1974) (see Leibbrandt 1975, Taylor 1976, Slavnov and Faddeiev 1978, Zavialov 1979 and Gottlieb and Donohue 1979 for further references). Over the past ten years the dimensional regularisation has proved to be an essential tool in QFT. Indeed it not only preserves Slavnov–Taylor identities but also avoids infrared problems which appear when subtractions are carried out at zero momentum. It has also been proved Speer (1976) that, under some conditions, the dimensional renormalisation is equivalent to the analytic renormalisation. The fact that within the dimensional renormalisation the Callan–Symanzik function $\beta(g)$ (Callan 1970, Symanzik 1971) is independent of the space–time dimension n (modulo $a(n-4)g$ factor) and also independent of mass ratios which appear in the theory, is an additional advantage of this renormalisation.

The dimensional regularisation procedure originated from a need to give a clear mathematical sense to the expression 'a divergent multidimensional integral' i.e. to define such a notion of such an integral that would allow standard operations of algebra and analysis to be performed correctly (e.g. multiplication, differentiation etc).

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It was found that it is just the theory of complex functions which allows us to give some precise mathematical meaning to the phenomenon of non-existence (within the measure theory) of such integrals. Roughly, the main step in the prescription for the dimensional regularisation of an arbitrary Feynman diagram consists in giving a meaning to the relevant integral $F(n)$

$$F(n)=\prod_{r=1}^{R}\int \mathrm{d}^n q_r (q_{l_1})_\lambda (q_{l_2})_\mu \ldots (q_{l_n})_\nu \prod_{i=1}^{L} (k_i^2-m_i^2+\mathrm{i}0)^{-1} \tag{1.1}$$

where L is the number of internal lines of the diagram, R the number of independent cycles and k_i are algebraic sums of q_r and external momenta p_i, for complex values of the parameter n. To achieve it one uses some parametric representation of the Green function that allows the integrations over q_i to be performed explicitly. This can be done e.g. with the aid of Feynman parameters (F-parameters) or the Schwinger representation (the α-representation), (we stick to the terminology used by Slavnov and Faddeiev (1978) p 158). If F-parameters are used, then one ends up with some integrals over Feynman's parameters which are intially defined for natural n only. The next step consists in giving a meaning either to a resulting explicit expression for $F(n)$ (if all integrations over F-parameters were performed explicitly) or directly to integrals over F-parameters, for complex values of n. Fortunately, the subtle difference between these two steps does not influence the physical results (see appendix 2 for a discussion of this point).

When the α-representation is used instead of F-parameters then the problem of an extension of $F(n)$ to complex n's is reduced to giving a meaning to the Gaussian integral

$$\tilde{G}(n)=\int \mathrm{d}^n q (2\pi)^{-n} \exp(-xq^2+2bq), \qquad x>0 \tag{1.2}$$

for complex values of n(b^μ and x do not depend on q^μ). It is of some importance to be understood that although both techniques yield exactly the same results they are only prescriptions, i.e. both are ambiguous. Since this fact is usually put aside in the physical literature (but see Butera *et al* 1974, Nouri-Moghadam and Taylor 1976, Zavialov 1979) we feel a few more words about such ambiguities may be needed. Let us consider the typical divergent integral one deals with in any regularisation scheme

$$I(n)=\int \mathrm{d}^n q (q^2-m^2+\mathrm{i}\varepsilon)^{-1}[(p-q)^2-M^2+\mathrm{i}\varepsilon]^{-1}. \tag{1.3}$$

Using Feynman parameters and performing some standard tricks (see e.g. 't Hooft and Veltman 1973, p 79) one gets

$$I(n)=\mathrm{i}\pi^w\Gamma(2-w)\int_0^1 \mathrm{d}x[-p^2x^2+(p^2-m^2+M^2)x+m^2]^{w-2} \tag{1.4}$$

where

$$w=\tfrac{1}{2}n, \qquad n\in N_1^\infty$$

and

$$\Gamma(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(K+z)}+\int_0^1 \mathrm{d}t\, t^{z-1}\,\mathrm{e}^{-t} \tag{1.5}$$

(see appendix 1 for further explanation of the symbols). The integration over x in (1.4) can be performed explicitly and the final result is

$$I(n) = \mathrm{i}\pi^{n/2} \exp(\mathrm{i}2\pi n) m^{2(n/2-2)} \Gamma(2-\tfrac{1}{2}n) F_1(1, 2-\tfrac{1}{2}n, 2-\tfrac{1}{2}n, 2; 1/x_1, 1/x_2) \tag{1.6}$$

where $F_1(a, b, c, d; u, v)$ is Appell's F_1-function (see appendix 2) and $x_{1/2} = \{M^2 - m^2 + p^2 \pm [(M^2 - m^2 + p^2)^2 + 4m^2p^2]^{1/2}\}/2p^2$. Now the problem of an extension of (1.6) to a domain of the complex n-plane must be set. Obviously the most desired extension would be just an analytic continuation. However, this is not the case unless some *ad hoc* additional assumption is made. To illustrate the point let us formulate the problem more precisely. The function $I(n)$ is defined by (1.6) for any natural n and one asks whether it is possible to find a unique interpolating function $\tilde{I}(z)$ of complex variable z, $z \in D \subset \mathbb{C}$, such that

$$\bigwedge_{z=n \wedge n \in N_1^\infty} [\tilde{I}(z) - I(n)] = 0. \tag{1.7}$$

The answer is, of course, known and negative, there exist infinite numbers of such interpolatory functions. Any functions of the type

$$\tilde{I}(z) = \mathrm{i}\pi^{z/2} \exp(\mathrm{i}2\pi z) m^{2(\frac{1}{2}z-2)} \Gamma(2-\tfrac{1}{2}z) F_1(1, 2-\tfrac{1}{2}z, 2-\tfrac{1}{2}z, 2; 1/x_1, 1/x_2)\varphi(z) + H(z)$$

where $\varphi(z)$ and $H(z)$ are periodic functions of z of period 1 such that $\varphi(0) = 1$ and $H(0) = 0$ (in general $H(z)$ does not vanish identically) can be considered as an interpolatory function for $I(n)$.

So one may wonder what necessary and sufficient conditions must be imposed upon an interpolatory function $\tilde{I}(z)$ to make a unique choice possible? Unfortunately, no such uniqueness conditions are known in the literature. The answer depends on the class of interpolatory functions one likes to deal with. Let us consider e.g. two interpolating functions $I_A(z)$ and $I_B(z)$ which obey (1.7) and let us form the difference $R(z) = I_A(z) - I_B(z)$. Obviously $R(n) = 0$, $n \in N_1^\infty$. If we assume further that e.g. $R(z)$ is regular and of exponential type in the half-plane $\mathrm{Re}(z) \geqslant 0$ and its indicator diagram is not bounded on the right by a vertical line segment of length 2π or more than $R(z) \equiv 0$ by virtue of the Carlson–Dufresnoy–Pisot theorem (Dufresnoy and Pisot 1951) (we have assumed additionally that $R(0) = 0$). So if one takes $I_A(z) = I(z)$ and $I_B(z)$ such that the pre-conditions of the CDP theorem are satisfied then $I(z)$ is the unique interpolatory function among the functions whose difference $R(z)$ satisfies the CDP theorem. However, one may use a lot of other theorems of such a type (see e.g. Boas 1954, Bieberbach 1955, Whittaker 1935) to introduce interpolating functions with different properties into the theory.

Similar problems also arise when one resorts to the α-representation instead of F-parameters. The ambiguity within the α-technique is related to the question of how one sets the problem of an extension of the Gaussian integral (1.2) to complex values of n. The conventional approach consists, of course, in assuming that

$$G(w) = \pi^w x^{-w} \exp(b^2/x) \qquad x > 0,\ w \in \mathbb{C} \tag{1.8}$$

where $G(w) \equiv \tilde{G}(2w)$. Although it is obvious that (1.8) is only the simplest extension and many others are, in principle, allowed, it is worth noting however, that a subtle difference exists between F-parameters and the α-representation techniques, concerning the question of the extension to the complex dimension of the resulting expressions. When F-parameters are used, ambiguities related to the non-uniqueness of the extension appear when one considers a new integral. This means that contributions from different Feynman graphs could, in principle, be extended in different ways, and we see no criterion for selection, as there is a virtually unlimited number of possible extensions, unless some *ad hoc* assumptions are made. The α-representation technique is free from such an unwanted abundance. Once the Gaussian integral $G(w)$ has been defined for $w \in \mathbb{C}$ then there is no room for further speculation. This probably explains why only the α-representation was used by Capper and Leibbrandt in their attempt to go beyond the conventional definition of $G(w)$. The latter authors have proposed to replace (1.8) by

$$\int d^{2w}q \exp(-xq^2+2bq) = \pi^w x^{-w} \exp[b^2/x - xf(w)] \qquad x>0 \qquad (1.9)$$

where $f(w)$ is an entire function which obeys the conditions:

(i) $f^{(k)}(w)=0$ for $w=n/2$, $n \in N_0^\infty$, $k \in N_0^{k_0}$, $k_0<\infty$ and $k_0 \geqslant 2$,

(ii) $\mathrm{Re} f(w)>0$ for any $w \neq n/2$ and some $\mathrm{Im} w \neq 0$ (Capper and Leibbrandt 1974a, b, c; Leibbrandt 1975).

Unfortunately, the question of a relationship between the conventional approach and the results obtained by those authors has never been settled. Although the Capper–Leibbrandt attempt has some drawbacks (see following sections) it nevertheless raises some interesting questions about the singularity structure of Feynman diagrams in the complex dimensional plane, such as:

(a) Is the conventional meromorphic structure (in the n-plane) pre-determined by the standard definition of the Gaussian integral $G(w)$ (see (1.8)) or can such a structure can be obtained by working with a non-conventional definition of $G(w)$?

(b) Is it possible to re-define the Gaussian integral in such a manner that one could get a more complicated singularity structure e.g. for a one-closed-loop contribution, a logarithmic singularity or, in general, non-algebraic singularity at the physical point $n=4$?

Since the singularity structure (in the n-plane) of a given Feynman graph is a pivot of the dimensional regularisation, the importance of such questions seems to be obvious. The aim of this paper is to examine the above questions. But it should be obvious from the beginning that such questions cannot be settled once for all or *ab ovo usque ad mala*. The first is due to the fact that one can invent as many non-conventional definitions of $G(w)$ as one wishes. The second is related to the unpleasant reality that technical troubles appear at such an early stage of development that one is forced to restrict considerations to lowest-order graphs (integrations can usually be performed explicitly for lowest-order Feynman diagrams only).

Thus any such answer would always be only partial and this is an inherent feature of the problem. Although the answers we get are always partial, nevertheless we think it is of some interest and makes for a better understanding of the dimensional regularisation technique to examine the singularity structure (in the complex dimensional plane) of the lowest-order graphs that one obtains by employing a non-conventional definition of the Gaussian integral. We would like to point out that we do not propose a new regularisation procedure for Feynman diagrams. The main aim of this paper is to demonstrate that a Capper–Leibbrandt (CL) type approach to the question of the dimensional regularisation introduces more problems than it solves.

Throughout the paper we employ the standard α-representation for propagators and suppose as usual that

$$\int \ldots \int \mathrm{d}^{2w}q_1 \ldots \mathrm{d}^{2w}q_L \int_0^\infty \ldots \int \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_K \ldots$$

$$= \int_0^\infty \ldots \int \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_K \ldots \mathrm{d}^{2w}q_1 \ldots \mathrm{d}^{2w}q_L \ldots .$$

The last equality, combined with a definition of the Gaussian integral in a complex dimensional w-plane, serves, in fact, as a formal definition of an integral in complex dimension w which one deals with in a dimensional regularisation scheme (see e.g. Taylor 1976, Slavnov and Faddeiev 1978, Zavialov 1979). Whether such a definition is sufficient to ensure a self-consistency of the schemes we consider in the paper, is unclear to us. It is known however that the standard dimensional regularisation scheme is internally consistent only if certain additional assumptions are made e.g. a definition equating massless tadpole integrals to zero (but see Breitenlohner and Maison (1977 p 73) for a discussion of IR-renormalised Feynman amplitudes with tadpole terms). In the cases we deal with in the paper, there seems to be no necessity for the introduction of such *ad hoc* definitions for lowest-order Feynman amplitudes at least. Since our primary aim was to examine a practical importance of the CL type schemes and it turned out that they were unsuitable for application, we think that additional investigations, in order to check whether the schemes have more internal faults, are unnecessary. So we take the stand that the question of self-consistency of the CL type schemes is beyond the scope of this paper.

The outline of the paper is as follows. In § 2 we introduce a one-parameter family of definitions of the Gaussian integral in the complex w-plane. Then we evaluate the integral $\int \mathrm{d}^{2w}q/(2\pi)^{2w}\,(q^2)^z$, $z \in \mathbb{C}$, and discuss the 't Hooft–Veltman hypothesis, tadpole and $\delta^4(0)$ terms. The central feature is the discussion of one closed loop and some integral associated with the pure graviton triangle diagram for different values of N in § 3. The basic notation is explained in appendix 1 where we also evaluate the integral

$$\int_0^\infty \mathrm{d}x\, x^{z-1}\{\exp[-x^N f(z)]\}_p F_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| -sx\right)$$

for $N \in R^1 - (0, 1/B)$. In appendix 2 we prove that the integral $I(w)$ (see (1.4)) has a zero-mass limit and point out conditions under which this limit coincides with the relevant massless integral. We consider only the simplest scalar theory throughout the paper and no further reference to this proviso will be made.

2. Gaussian integral, the 't Hooft–Veltman hypothesis, tadpole integral and the $\delta^4(0)$ term.

Prior to postulating a particular form of an interpolatory function for the Gaussian integral $G(n)$ (see (1.2)) we remind the reader of the criterion that any such function must obey. One requires that any interpolatory function $G(w)$ satisfies the conditions:

(a) $$\bigwedge_{w \in N_1^\infty} [G(w) - \pi^w x^{-w} \exp(b^2/x)] = 0, \qquad x > 0 \qquad (2.1)$$

(b) there should exist a domain in the complex w-plane such that a sum of contributions F_g from all Feynman diagrams of a given order is an analytic function of w, $w \in D \subset \mathbb{C}$; one requires also that there should exist a path K such that an analytic continuation of F_g from D to some neighbourhood O_2 of the point $w = 2$ is possible along K (excluding, in general, the point $w = 2$).

The first requirement is obvious and the second is necessary for there to be a theory at all.

Now let us consider a one-parameter family of interpolating functions $G_N(w)$ for the Gaussian integral $G(n)$

$$G_N(w) \equiv \int d^{2w}q \exp(-xq^2 + 2bq) = \pi^w x^{-w} \exp[(b^2/x) - x^N f(w)]$$
$$n = 2w, \qquad w \in \mathbb{C}, \qquad N \in R^1, \qquad |N| < \infty, \qquad x > 0 \tag{2.2}$$

where although the vector b^μ is defined initially only in a natural dimensional space, the scalar product bq is also well defined for complex w (see Bergere and David (1979) for a definition of the scalar product in dimension w); x is a c-number and $f(w)$ is any function satisfying the requirements (i) and (ii) given in § 1 (see (1.9)). One obtains the conventional interpolatory function $G_0(w)$ by taking $N = 0$ (the remaining factor $\exp[-f(w)]$ is unimportant). The CL interpolating function (1.9) is just $G_1(w)$.

Each function $G_N(w)$ satisfies, of course, the condition (a). However, it is not obvious that all $G_N(w)$ satisfy the requirement (b). We will give some arguments in § 3 that it may not be in the case for N being negative irrationals. Unfortunately, a general case is much more complicated than the relevant standard case (see Ashmore (1973, theorems 2.1–2.2) and also Speer (1969, theorems 2.17, 3.4 and 3.8)) and will not be discussed herein.

The choice of (2.2) as interpolating functions for $G(n)$ is, of course, arbitrary, but the degree of such an arbitrariness is no greater and no less than that in setting $f(w) \equiv 0$ as one usually does. When $N \neq 0$ then all interpolatory functions $G_N(w)$ have some drawbacks. The most important one is that, except at the point $w = 2$, regularisation schemes based on (2.2) preserve Slavnov–Taylor (ST) identities only up to a given and fixed order of perturbation theory determined by k_0 (see (1.9)). This means that any such regularisation scheme would occupy an intermediate position, e.g., the analytic regularisation which does not preserve ST identities and the standard dimensional regularisation that fully preserve them. Another shortcoming is that

$$\partial_x G_N(w) \neq \frac{\partial}{\partial b_\mu} \frac{\partial}{\partial b_\mu} G_N(w), \qquad N \neq 0$$

whereas

$$\partial_x G_0(w) = \frac{\partial}{\partial b_\mu} \frac{\partial}{\partial b_\mu} G_0(w).$$

This is rather a minor technical drawback but it hinders the application of some useful computational tricks.

The advantages of (2.2) over some other possible non-conventional choices for $G(w)$ seem to be its second-to-none simplicity and the fact that it allows calculations of the contribution from the scalar one-closed-loop to be performed explicitly (barring the interval $0 < N < 1$). So some comparison of our result with the results already known is possible and this greatly facilitates discussion.

Since the CL interpolatory function $G_1(w)$ has been applied for calculating only massless integrals (see Capper and Leibbrandt 1974a, b, c) we will discuss also only the massless case to get a comparison with that result (the treatment of relevant massive cases is obvious and need not be discussed here).

In the rest of this section we will assume that $N \in R^1 - \{0\}$. Since for any fixed N equation (2.2) defines, in fact, not one function in the w-plane but the set of unconnected functions (due to the factor $\pi^w x^{-w}$), we suppose that one of them is picked up and remains fixed hereafter.

Now we will check whether the integral

$$V(n) = \int \frac{d^{2w}q}{(2\pi)^{2w}} (q^2)^n \tag{2.3}$$

vanishes for all complex w and all *finite* non-negative integers n when the definition (2.2) is used for calculation. Multiplying both sides of the equality

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} \exp(-xq^2) = (4\pi x)^{-w} \exp[-x^N f(w)] \tag{2.4}$$

by x^{z-1} and subsequently integrating over x one obtains

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} \int_0^\infty dx\, x^{z-1} \exp(-xq^2) = (4\pi)^{-w} \int_0^\infty dx\, x^{z-w-1} \exp(-fx^N)$$

hence

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} (q^2)^{-z} = (4\pi)^{-w} \frac{\operatorname{sgn} N}{N} f^{(w-z)/N} \Gamma\left(-\frac{w-z}{N}\right) \Big/ \Gamma(z) \tag{2.5}$$

$$\operatorname{Re}(z-w)/N > 0, \qquad \operatorname{Re} z < 0 \qquad |\arg f| < \pi/4 \quad \text{or} \quad |\arg f| = \pi/4$$

$$\text{if } 0 < xe(z-w)/N < 1 \tag{2.5a}$$

(see e.g. Luke (1969) 2.1.(1)).

It is easy to check that no envelope of holomorphy exists for the right-hand-side of (2.5). This means that (2.5) may be analytically continued outside the domain (2.5a) so (2.5) is well defined as an analytic function of two complex variables (w, z) in $\mathbb{C}^2$ and one may rewrite (2.5) as

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} (q^2)^{z} = (4\pi)^{-w} \frac{\operatorname{sgn} N}{N} f^{(w+z)/N} \Gamma\left(-\frac{w+z}{N}\right) \Big/ \Gamma(-z). \tag{2.6}$$

Inserting $z = n$, $n \in N_0^\infty$ into (2.6) one gets $V(n) = 0$ for all complex w and finite non-negative integers n, i.e. the 't Hooft–Veltman hypothesis holds also when the Gaussian integral is defined by (2.2) and $N \neq 0$. The proof fails when $N = 0$. When $z = -n$ then (2.6) yields

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} (q^2)^{-n} = (4\pi)^{-w} \frac{\operatorname{sgn} N}{N} f^{(w-n)/N} \Gamma\left(-\frac{w-n}{N}\right) \Big/ \Gamma(n) \tag{2.7}$$

and for $n = 1$ and $N = 1$ one obtains from (2.7) the result obtained previously by Capper and Leibbrandt (1974a). The last case may be calculated directly with the help of (2.2) and the conventional representation for $(q^2)^{-1}$

$$(q^2)^{-1} = \int_0^\infty dx \exp(-xq^2), \qquad q^2 > 0$$

namely one gets

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}(q^2)^{-1} = \int_0^\infty dx \int \frac{d^{2w}q}{(2\pi)^{2w}} \exp(-xq^2)$$
$$= (4\pi)^{-w}\int_0^\infty dx\, x^{-w} \exp(-fx^N)$$
$$= (4\pi)^{-w}|N|^{-1}f^{(w-1)/N}\Gamma[(1-w)/N]. \tag{2.8}$$

The last integral has been calculated for $\mathrm{Re}(1-w)/N > 0$. However, being given in terms of analytic functions of w it can be analytically continued from the domain $\mathrm{Re}(1-w)/N > 0$ to the whole complex w-plane. The $\delta^4(0)$ terms which appear, e.g. in theories which contain two or more derivatives in a nonlinear Lagrangian, may be formally replaced by the integral $\int d^{2w}q/(2\pi)^{2w}$ within the dimensional regularisation. That integral can be calculated explicitly either be invoking equation (2.6) for $z=0$ or by performing the trick

$$\int \frac{d^{2w}q}{(2\pi)^{2w}} = \int \frac{d^{2w}q}{(2\pi)^{2w}}\frac{q^2}{q^2}$$
$$= -\int_0^\infty dx\,\partial_x \int \frac{d^{2w}q}{(2\pi)^{2w}} \exp(-xq^2)$$
$$= -(4\pi)^{-w}\int_0^\infty dx\,\partial_x[x^{-w}\exp(-fx^N)] = 0.$$

So the last equation is consistent with (2.6) and the $\delta^4(0)$ vanishes identically as it occurs within the CL scheme too. It is worth noting that apart from an 'obvious' equality

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}\int_0^\infty dx\,g(x)\exp(-xq^2) = (4\pi)^{-w}\int_0^\infty dx\,g(x)x^{-w}\exp(-fx^N) \tag{2.9}$$

there also exists another equality that can serve as a 'source' of formal definitions of integrals in the complex dimension. Namely, it is an equation one gets by applying to both sides of (2.4) an operator of fractional derivative D_x^α (see e.g. Oldham and Spanier 1974) and subsequently integrating over x

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}\int_0^\infty dx\,g(x)D_x^\alpha \exp(-xq^2)$$
$$= (4\pi)^{-w}\int_0^\infty dx\,g(x)D_x^\alpha[x^{-w}\exp(-fx^N)] \qquad \alpha\in\mathbb{C} \tag{2.10}$$

i.e.

$$\frac{1}{\Gamma(1-\alpha)}\int \frac{d^{2w}q}{(2\pi)^{2w}}\int_0^\infty dx\,g(x)x^{-\alpha}{}_1F_1\left(\begin{matrix}1\\1-\alpha\end{matrix}\middle| -xq^2\right)$$
$$= \int_0^\infty dx\,g(x)D_x^\alpha[x^{-w}\exp(-fx^N)], \qquad \mathrm{Re}\,\alpha \geqslant 0 \tag{2.11}$$

where $g(x)$ is a given function of x. The last equality follows from (2.10) and the relations

$$D_x^\alpha x^a = \frac{\Gamma(a+1)}{\Gamma(a+1-\alpha)} x^{a-\alpha}$$

$$D_x^\alpha \exp(-bx) = D_x^\alpha \sum_{n=0}^{\infty} (-b)^n x^n/(1)_n$$

$$= \sum_{n=0}^{\infty} (-b)^n \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}/(1)_n$$

$$= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \sum_{n=0}^{\infty} \frac{(1)n}{(1-\alpha)_n} (-bx)^n/(1)_n$$

$$= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} {}_1F_1\left(\begin{matrix} 1 \\ 1-\alpha \end{matrix} \middle| -bx \right) \qquad \mathrm{Re}\, \alpha \geqslant 0.$$

Attempts to calculate the right-hand side of (2.11) for some trial functions $g(x)$ quickly lead to conclusions that there seems to exist no universal form for $D_x^\alpha[x^{-w}\exp(-fx^N)]$ which would serve our purpose for an arbitrary $g(x)$. So we will give below three different forms for $D_x^\alpha[x^{-w}\exp(-fx^N)]$ that seem to be useful for future applications. The equality

$$D_x^\alpha[x^{-w}\exp(-fx^N)] = x^{-(w+\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(1-w+nN)}{\Gamma(1-w+nN-\alpha)} (-fx^N)^n/(1)_n \qquad \mathrm{Re}\, \alpha \geqslant 0 \tag{2.12}$$

which holds for an arbitrary $N \in \mathbb{C}$ may be rewritten for $N > 0$ in the form

$$D_x^\alpha[x^{-w}\exp(-fx^N) = x^{-(w+\alpha)} H_{1,2}^{1,1}\left(fx^N \middle| \begin{matrix} (N, w) \\ (1, 0); (w+\alpha, N) \end{matrix} \right) \tag{2.13}$$

where the Fox H-function is defined by the contour integral

$$H_{p,q}^{m,n}\left(z \middle| \begin{matrix} (\alpha_1, a_1), \ldots, (\alpha_n, a_n); (\alpha_{n+1}, a_{n+1}), \ldots, (\alpha_{n+p}, a_{n+p}) \\ (\beta_1, b_1), \ldots, (\beta_m, b_m); (\beta_{m+1}, b_{m+1}), \ldots, (\beta_{m+q}, b_{m+q}) \end{matrix} \right) = \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}s\, h(-s) z^{-s}. \tag{2.13a}$$

Further

$$h(s) = \frac{\prod_{j=1}^{n} \Gamma(1-a_j+\alpha_j s) \prod_{r=1}^{m} \Gamma(b_r - \beta_r s)}{\prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \prod_{r=m+1}^{q} \Gamma(1-b_r+\beta_r s)} \tag{2.13b}$$

$$0 \leqslant n \leqslant p, \qquad 1 \leqslant m \leqslant q; \qquad \alpha_i (i = 1, \ldots, p), \qquad \beta_k (k = 1, \ldots q)$$

are positive numbers and C runs from $s = \infty + \mathrm{i}k$ to $s = \infty - \mathrm{i}k$. Here k is a constant with $k > |\mathrm{Im}\, b_r/\beta_r|$ $(r = 1, \ldots, m)$ (see Braaksma (1963) for further restrictions on $\alpha_1, \beta_s, a_i, b_s$). For the case we deal with (see (2.13))

$$h(s) = \frac{\Gamma(1-w+Ns)}{\Gamma(1-w-\alpha+Ns)} \Gamma(-s)$$

and w is a complex number such that $\lambda + 1 - w + \nu N \neq 0$, λ, ν, $\lambda = 0, 1, 2, \ldots,$ and C is a contour such that the points $s = -\lambda (s = (\nu + 1 - w)/N)$ lie to the right (left) of C respectively, while further C runs from $s = \infty + \mathrm{i}k$ to $s = \infty - \mathrm{i}k$, $k > 0$. The relation

(2.13) is especially useful whenever $g(x)=\Sigma_1 d_i x^{c_i}$ since, in such a case, the integral on the right-hand side of (2.10) is simply a sum of Mellin transforms of $H^{1,1}_{1,2}$ and can be calculated immediately with help of the well known formula

$$\int_0^\infty \mathrm{d}x\, x^{z-1} H^{m,n}_{p,q}\left(x \middle| \begin{matrix}(\alpha_p, a_p)\\(\beta_q, b_q)\end{matrix}\right) = h(-z) \tag{2.14}$$

where $H^{m,n}_{p,q}$, $h(s)$ are defined by (2.13a), (2.13b) respectively. Since (2.13) does not hold for negative or complex N's then we will use the generalised Leibnitz rule

$$D^\alpha_x[g(x)h(x)] = \sum_{n=0}^\infty \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)\Gamma(n+1)}(D^n_x g)D^{\alpha-n}_x h$$

to calculate $D^\alpha_x[x^{-e}\exp(-fx^N)]$ for $N\in\mathbb{C}$. Simple calculations yield

$$D^\alpha_x[x^{-w}\exp(-fx^N)] = \frac{\Gamma(1-w)}{\Gamma(1-w-\alpha)}\sum_{n=0}^\infty (-1)^n \frac{(-\alpha)_n}{(1-w-\alpha)_n}\frac{x^{n-w-\alpha}}{(1)_n} D^n_x \exp(-fx^N). \tag{2.15}$$

When $N=-K$, $K\in N^\infty_0$ then (2.12) may be rewritten as

$$D^\alpha_x[x^{-w}\exp(-fx^N)] = x^{-w-\alpha}\frac{\Gamma(1-w)}{\Gamma(1-w-\alpha)}\,{}_KF_K\left(\begin{matrix}a_K\\b_K\end{matrix}\middle| -fx^N\right) \tag{2.16}$$

where $b_r=(w+r-1)/K$, $a_r=b_r+\alpha/K$, $r=1,2,\ldots,K$.

Assuming $g(x)=x^a$ in (2.11) and using equations (2.13) and (2.14) one may check that for $N>0$ the equality (2.6) follows from (2.10) too. Similarly (2.11) and (2.16) yield (2.6) when $N=-K$ and $g(x)=x^a$ (see Luke 1969, 3.6 (17)).

Finally we calculate the integral $\int \mathrm{d}^{2w}q/(2\pi)^{2w}(q^2+2pq+a)^{-z}$, $z\in C$. Substituting $b=-xp$ into (2.2) one gets

$$\int \frac{\mathrm{d}^{2w}q}{(2\pi)^{2w}}\exp[-x(q^2+2pq+a)] = (4\pi)^{-w}x^{-w}\exp(xp^2-xa-fx^N).$$

Multiplying both sides of the last equality by x^{z-1} and integrating over x one finds

$$\int \frac{\mathrm{d}^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{-z} = \frac{(4\pi)^{-w}}{\Gamma(z)}\int_0^\infty \mathrm{d}x\, x^{z-w-1}\exp[-(ax-p^2x+fx^N)]$$

and

$$\int \frac{\mathrm{d}^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{-z} = U(w,z) \tag{2.17}$$

where either

$$U(w,z) = \frac{(4\pi)^{-w}}{\Gamma(z)}(a-p^2)^{w-z}\sum_{n=0}^\infty\left[\frac{f}{(p^2-a)^N}\right]^n \Gamma(z-w+nN)/(1)_n \tag{2.17a}$$

$$N<1 \vee N=1 \wedge |f/(p^2-a)|<1$$

or

$$U(w,z) = \frac{(4\pi)^{-w}}{\Gamma(z)}\frac{\operatorname{sgn} N}{N}\sum_{n=0}^\infty (p^2-a)^n f^{(w-z-n)/N}\Gamma\left(\frac{n+z-w}{N}\right)\Big/(1)_n \tag{2.17b}$$

$$\frac{1}{N}<1 \vee N=1 \wedge |(p^2-a)/f|<1.$$

It is easy to see that (2.17a) yields, for $N=0$, the well known formula (apart from an inessential factor $\exp f$) (see e.g. Leibbrandt (1975, A1))

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{-z} = \frac{(4\pi)^{-w}}{\Gamma(z)}(a-p^2)^{w-z}\Gamma(z-w)\exp f \qquad (2.17c)$$

and for $N=1$ one obtains from (2.17a)

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{-z} = \frac{(4\pi)^{-w}}{\Gamma(z)}(a-p^2)\Gamma(z-w){}_1F_0(z-w; f/(p^2-a))$$

$$= (4\pi)^{-w}(f+a-p^2)^{w-z}\Gamma(z-w)/\Gamma(z) \qquad (2.18a)$$

(see e.g. Luke 1969, 6.2.1(2)). Similary (2.17b) yields for $N=1$

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{-z} = \frac{(4\pi)^{-w}}{\Gamma(z)}f^{w-z}\Gamma(z-w){}_1F_0(z-w; (p^2-a)/f)$$

$$= (4\pi)^{-w}(f+a-p^2)^{w-z}\Gamma(z-w)/\Gamma(z) \qquad (2.18b)$$

so for $N=1$ one gets the unique formula for $U(w,z)$.

It is obvious now that for integrals of a type (2.17) the CL regularisation (Capper and Leibbrandt 1974a) consists in shifting a mass square $a \to a+f(w)$ i.e., it is equivalent to an introduction of an additional complex mass at the intermediate stage of calculations. One may expect then that such a procedure will lead to troubles with gauge invariance of the theory and this is indeed the case.

It is worth noting that, multiplying both sides of the equality

$$\int d^{2w}q(2\pi)^{-w}\exp[-x(q^2+2pq+a)] = (4\pi)^{-w}x^{-w}\exp[-x(a-p^2+f)],$$

(i.e. $N=1$), by $x^{-z}G^{m,n}_{p,q}(sx|^{a_p}_{b_q})$ and integrating over x one obtains

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}(q^2+2pq+a)^{z-1}G^{m,n+1}_{p+1,q}\left(\frac{s}{q_2+2pq+a}\middle|\begin{matrix} z,ap \\ b_q\end{matrix}\right)$$

$$= (4\pi)^{-w}s^{w+z-1}G^{m,n}_{p,q}\left(\frac{s}{a-p^2+f}\middle|\begin{matrix}1,a_p-w-z \\ bq-w-z\end{matrix}\right),$$

whereas assuming $g(x) = G^{m,n}_{p,q}(sx|^{a_p}_{b_q})$ one gets from (2.11) for $f(w)\equiv 0$

$$\int \frac{d^{2w}q}{(2\pi)^{2w}}G^{1+n,1+m}_{1+q,2+q}\left(\frac{q^2}{s}\middle|\begin{matrix}0, \alpha-b_1, \dots, \alpha-b_q \\ 0, \alpha-a_1, \dots, \alpha-a_p, \alpha\end{matrix}\right)$$

$$= (4\pi)^{-w}\frac{\Gamma(1-w)}{\Gamma(1-w-\alpha)}s^w\prod_{i=1}^{m}\Gamma(b_i+1-w-\alpha)\prod_{r=1}^{n}(w+\alpha-a_r)$$

$$\times\left\{\prod_{i=m+1}^{q}\Gamma(w+\alpha-b_i)\prod_{r=n+1}^{p}\Gamma(1+a_r-w-\alpha)\right\}^{-1}$$

(see Luke 1969, 5.6.1(1), 5.6.2(1), 5.6.3.(1)). Integrals of this type may be useful in theories with non-polynomial Lagrangians where some Green functions are related to Meijer's function $G^{m,n}_{p,q}$. The main result of this section is that neither the 't Hooft–Veltman hypothesis nor the tadpole integral can serve as a filtering device for an elimination of some N's.

3. One-loop graph

The examples we have considered in the previous section are inconclusive concerning the question of the choice of values that the parameter N can take to obtain, in principle, a reasonable regularisation technique. Obviously, other graphs have to be examined to get a more-or-less definite answer to our question. We now turn to the simplest one-closed-loop integral

$$I_{\mathrm{SE}}=\int \mathrm{d}^{2w}q(q^2)^{-1}[(p-q)^2]^{-1} \tag{3.1}$$

Using the conventional α-representation

$$\frac{1}{q^2}=\int_0^\infty \mathrm{d}x\,\exp(-xq^2), \qquad \frac{1}{(p-q)^2}=\int_0^\infty \mathrm{d}y\,\exp[-y(p-q)^2] \tag{3.2}$$

where $q^2>0$ and $(p-q)^2>0$ one finds that

$$I_{\mathrm{SE}}=\pi^w\int_0^\infty \mathrm{d}x\int_o^\infty \mathrm{d}y\,\exp(-yp^2)\int \mathrm{d}^{2w}q\,\exp[-(x+y)q^2+2y(pq)]\,. \tag{3.3}$$

Applying (2.2) one obtains

$$I_{\mathrm{SE}}=\pi^w\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y(x+y)^{-w}\exp\Big(-yp^2+\frac{p^2y^2}{x+y}-(x+y)^Nf(w)\Big) \tag{3.4}$$

and introducing new variables $v=x+y$, $uv=y$ one gets

$$I(N)=\pi^w\int_0^\infty \mathrm{d}u\,u^{1-w}\exp(-u^Nf)\int_0^1 \mathrm{d}v\,\exp[-p^2uv(1-v)] \tag{3.5}$$

where $I(N)=I_{\mathrm{SE}}$ and $f=f(w)$.

Since our main aim in this section is an examination of the N-dependence of $I(N)$ we will calculate first the integral

$$K(u)=\int_0^1 \mathrm{d}v\,\exp[-p^2uv(1-v)] \tag{3.6}$$

which does not depend on N.

Dividing the interval of integration into two intervals—$(0,\frac{1}{2})$ and $(\frac{1}{2},1)$—and introducing the new variable $t^2=v(1-v)$ one obtains

$$\begin{aligned}K(u)&=\int_0^2 \mathrm{d}t\,2t(1-4t^2)^{-1/2}\exp(-bt^2)-\int_{1/2}^0 \mathrm{d}t\,2t(1-4t^2)^{-1/2}\exp(-bt^2)\\&=4\int_0^{1/2}\mathrm{d}t\,t(1-4t^2)^{-1/2}\exp(-bt^2)\end{aligned} \tag{3.7}$$

where $b=p^2u$. After some trivial manipulations one has

$$K(u)=\frac{1}{2}\int_0^1 \mathrm{d}x(1-x)^{-1/2}\exp(-bx/4)=\frac{1}{2}B(\tfrac{1}{2},1)\int_0^1 \mathrm{d}x\,\exp(-bx/4)x^{a-1}(1-x)^{c-a-1} \tag{3.8}$$

where $a = 1$, $c = \frac{3}{2}$, i.e.

$$K(u) = {}_1F_1(1, \tfrac{3}{2}; -\tfrac{1}{4}b) \tag{3.9}$$

(see e.g. Luke 1969, 4.2(1)).

Inserting (3.9) into (3.5) we get our basic formula for $I(N)$

$$I(N) = \pi^w \int_0^\infty \mathrm{d}u\, u^{(2-w)-1}\, {}_1F_1(1, \tfrac{3}{2}; -p^2u/4) \exp(-u^N f) \tag{3.10}$$

which is up to the factor π^w, the Mellin transform of the function ${}_1F_1(1, \frac{3}{2}; -p^2u/4) \exp(-u^N f)$.

First we calculate $I(0)$ to check whether (3.10) yields the well known result which one obtains by putting $N = 0$ into the right-hand side of (2.2) and *eo ipso* into (3.5). When $N = 0$ the right-hand side of (3.10) is the well known integral (see e.g. Erdelyi 1954, 6.15.1(10)) and one has

$$\begin{aligned} I(0) &= \mathrm{e}^{-f}\pi^w 2^{2(2-w)}(p^2)^{w-2}\Gamma(2-w)\Gamma(w-1)\Gamma(\tfrac{3}{2})/\Gamma(w-\tfrac{1}{2}) \\ &= \mathrm{e}^{-f}\pi^w (p^2)^{w-2}\Gamma(2-w)\Gamma(w-1)\Gamma(w-1)/\Gamma[2(w-1)]. \end{aligned} \tag{3.11}$$

So equation (3.10), when $N = 0$, yields the conventional result which one obtains by inserting $N = 0$ into (3.5), for $N = 1$ the integral is also known and one finds

$$I(1) = \pi^w f^{w-2}\Gamma(2-w)\, {}_2F_1(1, 2-w; \tfrac{3}{2}; -p^2/4f) \qquad |4f| > |p^2| \tag{3.12}$$

(see e.g. Luke 1969, 3.6(13)).

Now we apply Kummer's theorem (see e.g. Erdelyi 1954, 2.1.4(44)) to ${}_2F_1(1, 2-w; \frac{3}{2}; -p^2/4f)$ and deduce that

$$I(1) = \pi^w \Gamma(2-w)(f + p^2/4)^{w-2}\, {}_2F_1(\tfrac{1}{2}, 2-w; \tfrac{3}{2}; p^2/(p^2+4f)) \tag{3.13}$$

$$2 > \operatorname{Re} w > 1 \qquad \operatorname{Re} f \geqslant 0.$$

Keeping in mind that ${}_2F_1(\frac{1}{2}, 2-w; \frac{3}{2}; 1) = 4^{w-2}\{\Gamma(w-1)\Gamma(w-1)/\Gamma[2(w-1)]\}$ when $\operatorname{Re} w > 1$, we rewrite (3.13) in a more convenient form

$$I(1) = \pi^w \Gamma(2-w) \frac{\Gamma(w-1)\Gamma(w-1)}{\Gamma[2(w-1)]} (p^2+4f)^{w-2}\, {}_2F_1[\tfrac{1}{2}, 2-w; \tfrac{3}{2}; p^2/(p^2+4f)]/{}_2F_1(1) \tag{3.14}$$

and

$$I(1) = I(0)(1 + 4f/p^2)^{w-2}\, {}_2F_1(\tfrac{1}{2}, 2-w; \tfrac{3}{2}; p^2/(p^2+4f))/{}_2F_1(1) \tag{3.15}$$

$$1 < \operatorname{Re} w < 2 \qquad \operatorname{Re} f \geqslant 0.$$

where ${}_2F_1(1) = {}_2F_1(\frac{1}{2}, 2-w; \frac{3}{2}; 1)$.

Performing an analytic continuation of the right-hand side of (3.14) to all such complex w so that it is well defined in terms of the theory of complex functions, one gets $I(1)$ in other areas of the w-plane. It is easy to see that (3.15) is well defined in some domains enclosing the points $w_k = \frac{1}{2}k$, $k \in N_0^\infty$. At these points

$$I(1) = \pi^w (p^2)^{w-2}\Gamma(2-w)\{\Gamma(w-1)\Gamma(w-1)/\Gamma[2(w-1)]\}, \qquad w = \tfrac{1}{2}n \qquad n \in N_0^\infty$$

so one sees that

$$I(0) = I(1) \qquad \text{when} \quad w = \frac{n}{2}, \quad n \in N_0^\infty.$$

Since the case $N=1$ is just the CL case (Capper and Leibbrandt 1974a) one sees that the CL procedure leads to the conventional singularity structure for (3.1) at least.

Now let us turn to the remaining cases (see appendix 1). When $N>1$ then

$$I(N)=\frac{1}{N}\sum_{n=0}^{\infty}(-p^2/4)^n f^{-(z+n)/N}\Gamma\left(\frac{z+n}{N}\right)\bigg/\left(\frac{3}{2}\right)_n \tag{3.16}$$

where $z=2-w$. Equation (3.16) implies that $I(N)$ has transcendental singularities at the points $w=\frac{1}{2}n$, $n\in N_0^{\infty}$. This means that $f(z)$ would appear in a renormalised Lagrangian. Since the conditions (i) and (ii) (see (1.9)) define a class $Y(f)$ of such functions instead of a particular function $f(z)$, such renormalised Lagrangians would be invariant under the transformation $f(w)\to f'(w), f(w)\in Y(f), f'(w)\in Y(f)$. Such an additional symmetry of Lagrangians seems to be artificial and lacks physical meaning. We consider it as a definite drawback of any such regularisation scheme. If such a scheme were adopted then a regularisation of a one-loop integral $I(N)$ would consist, in fact, in throwing away the integral. So the self-energy diagram would disappear tracelessly from the theory.

When N is a negative rational then the picture is different. The contribution from a one-closed-loop

$$I(N)=\frac{1}{K}\sum_{n=0}^{\infty}(-p^2/4)^n f^{(z+n)/K}\Gamma\left(-\frac{z+n}{K}\right)\bigg/\left(\frac{3}{2}\right)_n \qquad K=|N|, \qquad z=2-w \tag{3.17}$$

has a simple pole at the physical point $w=2$ as it also occurs in the conventional scheme (Speer and Westerwater 1971, 't Hooft and Veltman 1972, Bellini and Giambiagi 1972, Genta and Montaldi 1972, Ashmore 1972, 1973, Butera *et al* 1974, Speer 1974a, b, Vega and Schapošnik 1974). However, the behaviour of $I(N)$ at the points $w-\frac{1}{2}n$, $n\in N_0^{\infty}-\{2\}$ differs markedly from the relevant behaviour of $I(0)$. Let us choose e.g. $N=-1/A$, $A\in N_1^{\infty}$, one then finds

$$I(-1/A)=\pi^w A f^{(2-w)A}\Gamma[(w-2)A]\,{}_1F_{A+1}\left(\frac{1}{\frac{3}{2};\,(2-w+r/A)_A}\bigg| f^A p^2/4A^A\right) \tag{3.18}$$

hence $I(-1/A)=0$ when $w<2$ whereas $I(0)|_{w=1}$ has a simple pole (but $I(0)|_{w=1/2}=0$). If $w>2$ then $I(-1/A)$ has branch poles of high orders at points $w=l+\frac{1}{2}$, $l\in N_2^{\infty}$, and ordinary poles of high orders at $w=r+3$, $r\in N_0^{\infty}$ provided that Ak_0 is odd (if Ak_0 is even then $I(-1/A)$ has ordinary poles of corresponding order at $w=\frac{1}{2}$, $l\in N_3^{\infty}$). An ugly feature of the choice $N<0$ is a very sensitive dependence of the singularity structure of $I(N)$ on the choice of two arbitrary parameters N and k_0 (see (1.9)). Moreover, a natural tendency to preserve the gauge invariance of the theory up to a sufficiently high order of perturbation theory will lead to a really complicated singularity structure of $I(N)$. It has been pointed out in appendix 1 that although $I(N)$ can be calculated explicitly for negative irrational N's provided

$$w\in\left\{w:\ \bigwedge_{n\in N_1^{\infty}}(2+n-\mathrm{Re}\,w)/|N|\neq r,\quad r\in N_0^{\infty}\right\}$$

it is not known whether $I(N)$, as given e.g. (A1.4), is an analytic function of w in a domain of the w-plane. So such negative irrational N should be avoided. Anyhow, even if one were able (we were not) to prove that $I(N)$ given by (A1.4) is, in fact, an analytic function of w in a domain D also in this case, it would not change the situation. Such a regularisation scheme would still be very complicated.

The case $0<N<1$ remains most complicated. One cannot calculate explicitly $I(N)$ for such N's (see appendix 1) by one method and the singularity structure of $I(N)$ in the w-plane remains to be found. One may wonder what the difference is between $N=1$ and e.g. $N=-1$ cases for $I(N)$. It is easy to see that, by taking $N=1$, one changes the behaviour of the integrand (see (3.1)) for $u \to \infty$. This corresponds to changing, in general, infrared behaviour of the contribution from a one-closed-loop. By inserting $N=-1$ into (3.10) one modifies the ultraviolet behaviour of the corresponding diagram. Since there exists a domain D in the w-plane such that the integral $\int_0^\infty du\, u^{(2-w)-1}{}_1F_1(1, \frac{3}{2}; -p^2u/4)$ exists and is an analytic function of w, $w \in D$ then neither of the modifications is, in fact, necessary to achieve the goal of a regularisation scheme (for a one-closed-loop at least). It is obvious that this property is quite general: by taking $N \geqslant 1$ one modifies the infrared behaviour of I_{SE} (see (3.1)) and by choosing $N<0$ one changes the ultraviolet behaviour of I_{SE}. However, this is not the only difference between the two cases (i.e. $N \geqslant 1$ and $N<0$). Let us again consider the interpolatory functions $G_N(w)$ for $N=0$, $N=1$ and $N=-1/A$, $A \in N_1^\infty$. The functions $G_0(w)$ and $G_1(w)$ do have a common feature; in a sense, both lead to the same singularity structure in the w-plane of one-closed-loop: the integral (3.1) has only simple poles at any non-negative integer w. This is not the case when $N=-1/A$, $A \in N_1^\infty$; 'the degree of divergence' of the $I(-1/A)$ depends on the dimension of space. The higher $n \geqslant 4$ is taken, the larger is the order of the relevant pole; when $n<4$ all $I(-1/A)$ vanish, the first pole appears at $n=4$ and is a simple one. After n has passed the value, 4 poles of a higher order suddenly appear (we assume that Ak_0 is even) and the order of poles is strongly correlated with n—the dimension of space— and k_0 (which determines gauge invariant properties of a regularisation scheme fixed by A). The idea that the integral (3.1) could be more divergent as n increases does not seem to be unnatural. The fact that the conventional interpolatory function $G_0(w)$ (see (1.8)) yields for $I(0)$ the singularity structure which is, in a sense, n-independent when $n \geqslant 2$ ($I(0)$ has only simple poles at $n \in N_2^\infty$) seems to be a rather peculiar feature of dimensional regularisation schemes which were invented to cope mainly with infrared problems (e.g. the CL scheme (Capper and Leibbrandt 1974a)).

Now let us consider an example of the interpolating function for $G(n)$ which modifies both infrared and ultraviolet properties of the integrands in (3.1). Namely, let us suppose that

$$G(w)=\pi^w x^{-w} \exp[b^2/x-(x^N+x^{-N})f(w)] \tag{3.19}$$

and calculate I_{SE} (see (3.1)). It is easy to check that this time

$$\begin{aligned} I_{SE} &= \pi^w \int_0^\infty du\, u^{(2-w)-1}{}_1F_1(1, \tfrac{3}{2}; -p^2u/4) \exp[-(x^N+x^{-N})f(w)] \\ &= 2\frac{\pi^w}{N}\sum_{n=0}^\infty \frac{(-p^2/4)^n}{(\frac{3}{2})^n} K_{(2+n-w)/N}(2f), \qquad \operatorname{Re} f>0 \end{aligned} \tag{3.20}$$

where $K_\alpha(z)$ is a modified Bessel function (see e.g. Erdelyi (1954, 6.3 (17)). Analytic properties (in the w-plane) of (3.20) are rather complicated and there is no point in discussing them in any detail here. It is sufficient to point out that

$$K_0(x)=-J_0(x)\ln\frac{x}{2}+\frac{1}{2}\sum_{k=0}^\infty \left(\frac{x}{2}\right)^{2k} \psi(k+1)/\Gamma^2(k+1)$$

where $x \in R^1$

$$J_0(x) = \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} \Big/ \Gamma^2(l+1)$$

and $\psi(x)$ is the logarithmic derivative of $\Gamma(x)$ (see any textbook on Bessel functions), thus I_{SE} has a logarithmic singularity at least at the physical point $w=2$. Obviously interpolatory functions (3.19) are practically useless and mere curiosities.

One can generalise this conclusion to diagrams of higher order by considering the Schwinger parametric form of Feynman integrands and using (2.2) as interpolatory functions for the Gaussian integrals. It is obvious that by taking $N>1$ ($N<0$) one makes an *ad hoc* assumption that Feynman integrands are more infrared (ultraviolet) divergent than is assumed when (1.8) is substituted. One may, of course, be tempted to modify both ultraviolet and infrared behaviour of Feynman integrands by playing with a formula of the type (3.19). However, from the results obtained by Anikin *et al* (1980) it follows that a strong interplay exists between ultraviolet and infrared divergencies in massless theories. So it could happen, in principle, that regularisation schemes based on such interpolatory functions (3.19) would be internally inconsistent i.e. that catastrophic singularities in the w-plane (see Anikin *et al* (1980) for the explanation of this term) could appear in the theory. The standard interpolatory function (1.8) is most safe in this respect (a self-consistent scheme when both massive *and* massless particles are present has been constructed recently by Breitenlohner and Maison (1977), (see also Collins 1975, Bergere and David 1980) and interpolating functions of the type (3.19) seem to be most dangerous for self-consistency of the theory. To round up this section we consider the integral associated with the pure graviton diagram $\mathcal{J}_3 = \int d^{2w}k[k^2(k-p_2)^2(k+p_3)^2]^{-1}$.

After some calculations one gets with the help of (2.2)

$$\mathcal{J}_3 = \pi^w \int_0^\infty dz\, z^{2-w} \exp(-fz^N) \int_0^1 dx\, x \int_0^1 dy\, \exp(-zA)$$

where

$$p_1 = -(p_2+p_3)$$

$$A = x\{(1-x)[-\alpha y(1-y)+\beta(1-y)+\gamma y]+\alpha y(1-y)\}$$

$$\alpha = p_1^2, \qquad \beta = p_2^2, \qquad \gamma = p_3^2.$$

Let us assume further that $p_2 = -p_3$, $p_2^2 \neq 0$; one gets for this particular momentum configuration

$$\mathcal{J}_3 = \pi^w \int_0^\infty dz\, z^{(3-w)-1} \exp(-fz^N)\,{}_1F_1(1, \tfrac{3}{2}; -\beta z/4).$$

The last integral may be calculated easily (see appendix 1) and when $N=1$ one obtains

$$\mathcal{J}_3 = \pi^w \Gamma(3-w)(f+\beta/4)^{w-3}\,{}_2F_1(\tfrac{1}{2}; 3-w; \tfrac{3}{2}; \beta/(\beta+4f), \qquad p_2 = -p_3$$

so a singularity structure of J_3 at the physical point $w=2$ is determined entirely by the behaviour of ${}_2F_1$ at this point. Since it is known that if $c-a-b=0$ then ${}_2F_1(a, b; c; z)$ has a logarithmic singularity (see e.g. Klein 1933, p 18) then J_3 has a logarithmic singularity at $w=2$ when $p_2 = -p_3$ and $N=1$ (in fact ${}_2F_1(\frac{1}{2}, 1; \frac{3}{2}; z^2) = (1/2z)\log[(1+z)/(1-z)]$; see Slater 1966, (1.5.3)). This behaviour of J_3 at $w=2$

sharply constrasts with the standard pole singularity one gets when $N = 0$

$$\mathcal{J}_3(0) = \pi^w \int_0^\infty \mathrm{d}z\, z^{(3-w)-1}\,{}_1F_1(1, \tfrac{3}{2}; -\beta z/4)$$
$$= \pi^{w+1/2}(p_3^2/4)^{w-3}\Gamma(3-w)\Gamma(w-2)/2\Gamma(w-3/2).$$

Unfortunately, this fact remained unnoticed in Capper and Leibbrandt (1974a, b, c) and Leibbrandt (1975).

4. Conclusions

We have found that all non-conventional interpolatory functions for the Gaussian integral considered in this paper have some drawbacks concerning their applicability within a dimensional regularisation scheme. Although the 't Hooft–Veltman hypothesis, the tadpole integral (2.3*a*) and the $\delta^4(0)$ term are insensitive to a choice of N in (2.2), the one-loop integral (3.1) can serve as a filtering device for N.

We have demonstrated that when $N > 1$ then $G_N(w)$ leads, in general, to non-algebraic singularities and should be avoided. The CL interpolatory function $G_1(w)$ yields the usual pole structure for a one-loop integral at the physical piont. $w = 2$. However, $G_1(w)$ yields a very complicated momentum dependence for the integral J_3 related to the pure graviton triangle vertex, and for some momentum configuration $G_1(w)$, leads to a logarithmic singularity for J_3. We consider this property of $G_1(w)$ as definitely undesirable. The distinct feature of $G_N(w)$ taken for some negative rational N's (e.g. $N = -1/A$, $A \in N_1^\infty$) is that it is possible to correlate orders of poles which appear when considering $I(N)$ with the dimension of space (branch poles should be avoided anyhow). However, even in the simplest $N = -1$ case, the resulting singularity structure of $I(-1)$ is very complicated.

Our final conclusion is that since it is now known that, within the standard dimensional regularisation scheme, both massive and massless theories (the most important at least) can be treated in a consistent manner (see e.g. Breitenlohner and Miason 1977, Bergere and David 1980), a deviation of type (2.2) from the conventional interpolatory function $G_0(w)$, offers no significant improvement.

Appendix 1

The following basic notation is used throughout the paper: the set of all real numbers is denoted by R^1, $R_0^1 = R^1 - \{0\}$, the set of all non-negative integers is denoted by N_0^∞, a set $\{N_0^\infty - \{0, 1, 2, \ldots, k-1\}\}$ is denoted by N_k^∞. The set of all complex numbers is denoted by $\mathbb{C}$. In this appendix, the symbol $f(z)$ denotes an entire function of z such that $\mathrm{Re}\, f(z) > 0$. Sometimes we simply write f instead of $f(z)$ to make the formulae more compact and clear. We often employ the contracted notation for the generalised hypergeometrical function

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right)$$

(see e.g. Erdelyi 1953, 4.1.(1), or Luke 1969, 3.2(1)) and write it in the abbreviated forms either

$$ {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right) \quad \text{or} \quad {}_pF_q(a_p, b_q; z) $$

i.e.

$$ {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right) = {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}\middle| z\right) = {}_pF_q(a_p, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{(1)_n} \tag{A1.1} $$

When there is no possibility of confusion, we simply refer to (A1.1) as ${}_pF_q$. We assume that no denominator parameter b_i, $i = 1, \ldots, q$ is a negative integer or zero. We also put $\Gamma(a+n)/\Gamma(a) = (a)_n$. Furthermore we introduce the function $L(N, z, s, f)$ defined by equation (A1.2):

$$ L(N, z, s, f) = \int_0^\infty \mathrm{d}x\, x^{z-1} {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| -sx\right) \exp\left(-x^N f(z)\right) \tag{A1.2} $$

where $p = q + 1 - B$, $B \in \{1, 2, \ldots, q+1\}$, $N \in R^1$, $\operatorname{Re} f(z) > 0$. We often simply write $L(N)$ instead of $L(N, z, s, f)$ to simplify the notation. It is assumed throughout this appendix that, when $N < 0$, then parameters $(a_1, \ldots, a_p; b_1, \ldots, b_q; z; s)$ are taken such that the conditions under which the formula 3.6(17) given in Luke (1969, p 60) is valid are satisfied in our case too (the restriction $\operatorname{Re} z > 0$ is unnecessary when $N < 0$).

Since many more common special functions can be expressed in terms of ${}_pF_q$ we feel that an explicit form of $L(N, z, s, f)$ may be of some interest for mathematical physics. We were unable to find $L(N)$, except when $N = 0$ or $N = +1/A$ and $f(z) =$ constant, in the literature (see Erdelyi 1954, 4.23(19)) so we calculate it in this appendix. One can prove (Kalinowski *et al* 1980) for detailed proof:

(i) $N > 1/B$ or $N = 1/B$ and $|s/(Nf)^{1/N}| < 1$. If $N > 1/B \vee (N = 1/B) \wedge |s/(Nf)^{1/N}| < 1)$ then

$$ L(N) = \frac{1}{N} f^{-z/n} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} (-s/f^{1/n})^n \, \Gamma\left(\frac{z+n}{N}\right) \Big/ (1)_n \tag{A1.3} $$

and $L(N, z, s, f)$ is a multiple-valued analytic function of z. When $N = 1/A \wedge 1 \leqslant A \leqslant B \wedge A \in N_1^\infty$ then

$$ L(N) = A f^{-Az} \Gamma(Az)\, {}_{p+A}F_q\left(\begin{matrix} a_1, \ldots, a_p, z, z+1/A, \ldots, z+(A-1)/A \\ b_1, \ldots, b_q \end{matrix}\middle| -A^A f^{s/A}\right) \tag{A1.3a} $$

(ii) $N < 0$

If $N < 0$ and $z \in U$, $U = \{z\colon \wedge_{n \in N_1^\infty} (\operatorname{Re} z + n)/|N| \neq r, r \in N_0^\infty\}$ then

$$ L(N) = -\frac{1}{N} f^{-z/N} \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} (-s/f^{1/n})^n \, \Gamma\left(\frac{z+n}{N}\right) \Big/ (1)_n \tag{A1.4} $$

the last series converges whenever $z \in U$. If N is a negative rational then $L(N, z, s, f)$ given by (A1.4) is a multiple-valued analytic function of z. When $N = -1/A$, $A \in N_1^\infty$ then

$$ L\left(-\frac{1}{A}\right) = A f^{Az} (-Az)_p\, {}_pF_{q+A}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q, z+1/A, \ldots, z+(A-1)/A, z+1 \end{matrix}\middle| s f^A/A^A\right). \tag{A1.4a} $$

Appendix 2

In this appendix we will demonstrate that the zero-mass limit of $I(n)$ defined by (1.4) exists and under some additional (but trivial) conditions it coincides with the relevant massless integral.

Define

$$I_0(n) = \lim_{m^2\to 0} \lim_{M^2\to m^2} \frac{I(n)}{\mathrm{i}\pi^w \Gamma(2-w)}$$

then

$$I_0(n) = \lim_{m^2\to 0} (m^2)^{w-2} \exp(\mathrm{i}4\pi w)$$

$$\times F_1\left(1, 2-w, 2-w, 2; \frac{2}{1+(1+4m^2/p^2)^{1/2}}, \frac{2}{1-(1+4m^2/p^2)^{1/2}}\right)$$

$$= \exp(\mathrm{i}4\pi w) \lim_{m^2\to 0} (m^2)^{w-2} F_1(1, 2-w, 2-w, 2; 1, -p^2/m^2)$$

$$= \exp(\mathrm{i}4\pi w) \lim_{m^2\to 0} (m^2)^{w-2} \sum_{s=0}^{\infty} \frac{(1)_s(2-w)_s}{(2)_s(1)_s} {}_2F_1\left(1+s, 2-w, 2+s, -\frac{p^2}{m^2}\right)$$

(Appell and Kampe de Feriet 1926, ch I, equation (15), p 15). Now let us assume that Re $w = 2+\varepsilon$, $\varepsilon > 0$, to use safely Abel's lemma. Since

$${}_2F_1\left(1+s, 2-w; 2+s; -\frac{p^2}{m^2}\right) = \left(1+\frac{p^2}{m^2}\right)^{w-2} {}_2F_1\left(1, 2-w, 2+s; \frac{p^2}{p^2+m^2}\right)$$

and

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

(Erdelyi 1953, 2.9(4), 2.1(14)) then

$$I_0(n) = \exp(\mathrm{i}4\pi w)(p^2)^{w-2} \sum_{s=0}^{\infty} \frac{(2-w)_s\Gamma(w+s-1)}{\Gamma(w+s)}$$

$$= \exp(\mathrm{i}4\pi w)(p^2)^{w-2} \frac{\Gamma(w-1)\Gamma(w-1)}{\Gamma[2(w-1)]}.$$

So the zero-mass limit of the integral $I(n)$ exists and, up to the factor $\exp(\mathrm{i}4\pi w)$, coincides with the relevant massless integral. It is interesting to trace how the factor $\exp(\mathrm{i}4\pi w)$ appears. It is easy to see that this factor appears due to the fact that we defined the interpolatory function for $I(n)$ as

$$I(w) = \mathrm{i}\pi^w \Gamma(2-w) \int_0^1 \mathrm{d}x[-p^2x^2 + (p^2 - m^2 + M^2)x + m^2]^{w-2}, \qquad w \in \mathbb{C}$$

(see (1.4)). Had we first calculated this integral for $w = n/2$, $n \in N_2^\infty$, then the factor $\exp(\mathrm{i}4\pi w)$ disappears ($\exp(\mathrm{i}2\pi n) = 1$, $n \in N_0^\infty$). Obviously the presence of this factor does not influence the results for natural n's. However, it leads to some discrepancy for $n = (2k+1)/2$, $k \in N_0^\infty$. And we conclude that the best way to avoid such problems is to calculate $I(w)$ first for natural n and only then make an extension of the result to complex n's.

Finally we define Appell's F_1-function for the convenience of the reader:

$$\operatorname{Re}\alpha>0, \qquad \operatorname{Re}(\gamma-\alpha)>0$$

$$F_1(\alpha,\beta,\beta',\gamma;x,y)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 \mathrm{d}u\, u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'}$$

(see Appell and Kempe de Feriet 1926, ch II, equation (4), p 29). This function appears, in fact, quite often when one deals with problems of the dimensional regularisation it is easy to see that e.g. the integrals which appear in Cicuta *et al* (1980, equations (6) and (11)) can be expressed in terms of $F_1(\alpha,\beta,\beta',\gamma;x,y)$.

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